

DENSITIES NON-REALIZABLE AS THE JACOBIAN OF A 2-DIMENSIONAL BI-LIPSCHITZ MAP ARE GENERIC

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ABSTRACT. In this work, positive functions defined on the plane are considered from a generic viewpoint, both in the continuous and the bounded setting. By pursuing on constructions of Burago-Kleiner and McMullen, we show that, generically, such a function cannot be written as the Jacobian of a bi-Lipschitz homeomorphism.

1. INTRODUCTION

A direct consequence of the Fundamental Theorem of Calculus is that every positive continuous function ρ defined on a compact interval $[a, b]$ can be written as the derivative of a diffeomorphism, namely

$$f(x) := \int_a^x \rho(s) ds.$$

The very same formula shows that every positive L^∞ function that is bounded away from zero can be written as the a.e. derivative of a bi-Lipschitz homeomorphism.

As was shown by Burago-Kleiner [1] and McMullen [6] in very important works (see also [7]), this is no longer true in the 2-dimensional framework: there exist positive L^∞ (even continuous) functions that cannot be written as the Jacobian of a bi-Lipschitz homeomorphism of the plane. This pure analytical result is obtained as the fundamental step of another result of a discrete nature: there exist coarsely dense, uniformly discrete sets in the plane (that can be taken as subsets of \mathbb{Z}^2) that are not bi-Lipschitz equivalent to the standard lattice \mathbb{Z}^2 . Although recently a shortcut (and extension) to this last result has been produced in [2], the analytical one is interesting by itself, and deserves more attention. Based on Burago-Kleiner's version of this result (which is slightly more general than McMullen's), in this work we show that not only bad densities exist, but are also generic.

Main Theorem. A generic positive continuous function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ cannot be written as the Jacobian of a bi-Lipschitz homeomorphism. The same holds for a generic positive L^∞ function.

We point out that similar results hold (with the same proof) in dimension greater than 2. The restriction to the 2-dimensional case below just allows simplifying notation and computations.

2. THE BURAGO-KLEINER CONSTRUCTION

In this section, we review the Burago-Kleiner construction of a density that is nonrealizable as the Jacobian of a bi-Lipschitz map. We begin with some definitions and classical results.

Definition 1. A map $f : (X_1, d_1) \rightarrow (X_2, d_2)$ between two metric spaces is said to be *L-bi-Lipschitz* if for every pair of points $x, y \in X_1$, we have

$$\frac{1}{L} \cdot d_1(x, y) \leq d_2(f(x), f(y)) \leq L \cdot d_1(x, y).$$

We call the map f **bi-Lipschitz** if there exists some $L \geq 1$ such that f is L -bi-Lipschitz.

Definition 2. We say that two points $x, y \in \mathbb{R}^2$ are **A -stretched** under a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ if $\|f(x) - f(y)\| \geq A\|x - y\|$.

We remind a classical result, a proof of which may be found in an Appendix of [4].

Theorem (Rademacher). Let $U \subset \mathbb{R}^n$ be open and $f : U \rightarrow \mathbb{R}^m$ a Lipschitz map. Then f is almost everywhere differentiable in U , and its partial derivatives belong to L^∞ .

We denote by $I^2 := [0, 1] \times [0, 1]$ the standard unit square in \mathbb{R}^2 . The key tool for the existence of a *bad density*, that is, a density that is not realizable as the Jacobian of a bi-Lipschitz map, is given in the next proposition from [1].

Proposition 1. For any given $L > 1$ and $c > 0$, there exists a continuous function $\rho : I^2 \rightarrow [1, 1 + c]$ such that there is no L -bi-Lipschitz homeomorphism $\phi : I^2 \rightarrow \mathbb{R}^2$ satisfying

$$Jac(\phi) = \rho \text{ a.e.}$$

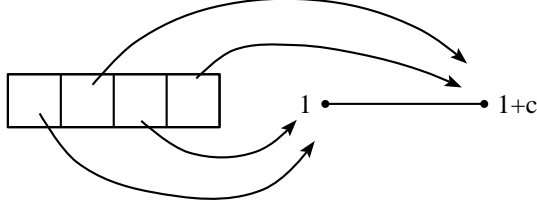
We next sketch the construction of the density above. First of all, Burago and Kleiner observe that it is sufficient to construct a measurable density $\rho_* : I^2 \rightarrow [1, 1 + c]$ having the property above. Indeed, if $(\rho_k)_{k \in \mathbb{N}}$ is a sequence of smoothings of the measurable function ρ_* that converges to ρ_* in L^1 , then by the Arzelà-Ascoli theorem, any sequence of L -bi-Lipschitz maps $\phi_k : I^2 \rightarrow \mathbb{R}^2$ such that $Jac(\phi_k) = \rho_k$ a.e will converge, up to a subsequence, to a bi-Lipschitz map $\phi : I^2 \rightarrow \mathbb{R}^2$ satisfying $Jac(\phi) = \rho_*$ a.e. See the proof of Proposition 2 for a variation of this argument.

The key tool for the construction of the measurable bad density is the **checkerboard function** $\rho_{N,c}$ (for $N = N(L, c)$ sufficiently large) defined below.

Definition 3. For $N \in \mathbb{N}$, consider the rectangle $R_N := [0, 1] \times [0, \frac{1}{N}]$ and the squares $S_i := [\frac{i-1}{N}, \frac{i}{N}] \times [0, \frac{1}{N}]$, with $i = 1, \dots, N$. Given $c > 0$, we define the function $\rho_{N,c} : R_N \rightarrow [1, 1 + c]$ by letting

$$\rho_{N,c}(x) = \begin{cases} 1 & \text{if } x \in S_i \text{ with } i \text{ even,} \\ 1 + c & \text{if } x \in S_i \text{ with } i \text{ odd.} \end{cases}$$

Using a quite long argument, Burago and Kleiner prove that for a large enough N , there exists $\epsilon = \epsilon(L, c) > 0$ such that for every L -bi-Lipschitz map $\phi : I^2 \rightarrow \mathbb{R}^2$ whose Jacobian $Jac(\phi)$ is equal to $\rho_{N,c}$ except for a set of measure less than ϵ , there exist two points $x, y \in R_N$ that are $(1 + \kappa)\|\phi(0, 0) - \phi(1, 0)\|$ -stretched under ϕ for a certain $\kappa = \kappa(L, c) > 0$. Using this, as a next step they modify the function on a rectangular neighborhood U of \overline{xy} by including a rescaled version of another checkerboard function. By the same argument, there are two points $x_2, y_2 \in U$ that are $(1 + \kappa)^2\|\phi(1, 0) - \phi(0, 0)\|$ -stretched by an homeomorphism realizing this new function as the Jacobian. Repeating this argument at smaller and smaller scales, we eventually obtain a function $\rho : R_N \rightarrow [1, 1 + c]$ which cannot be the Jacobian of an L -bi-Lipschitz map.


 FIGURE 1. The checkerboard function $\rho_{N,c} : R_N \rightarrow [1, 1+c]$.

The next lemma summarizes a slightly stronger version of the output of the construction above, and it also appears in [1]. It guarantees the non-existence of an L -bi-Lipschitz map with a certain prescribed Jacobian up to a small error by providing a finite collection of pairs of points that are stretched by a uniform amount under these maps.

Lemma 1. *Given $L > 1$ and $c > 0$, for each positive integer i there is a continuous function $\rho_i : I^2 \rightarrow [1, 1+c]$, a finite collection \mathcal{S}_i of non-intersecting segments $\overline{l_k r_k} \subset I^2$, and $\delta_i > 0$, with the following property: For every L -bi-Lipschitz map $f : I^2 \rightarrow \mathbb{R}^2$ whose Jacobian differs from ρ_i on a set of area less than δ_i , the endpoints of at least one of the segments from \mathcal{S}_i are $\frac{(1+\kappa)^i}{L}$ -stretched by f .*

3. THE CONTINUOUS CASE

Recall that a subset G of a metric space M is said to be **thick** if it contains a set of the form $\bigcap_{n \in \mathbb{N}} G_n$, where each G_n is an open dense subset of M .

Definition 4. *If all points of a thick subset have some property, then this is said to be a generic property of M .*

We now consider the space $C_+(I^2, \mathbb{R})$ of all positive continuous functions $\rho : I^2 \rightarrow \mathbb{R}$ with the norm $\|f\|_0 := \sup\{|f(x)| : x \in I^2\}$. In this section, we will use the Burago-Kleiner construction to prove the Main Theorem in the continuous setting.

Theorem 1. *Let \mathcal{C} be the set of all functions $\rho \in C_+(I^2, \mathbb{R})$ such that there is no bi-Lipschitz map $\phi : I^2 \rightarrow \mathbb{R}^2$ satisfying $\rho = \text{Jac}(\phi)$ a.e. Then \mathcal{C} is a thick subset of $C_+(I^2, \mathbb{R})$.*

This Theorem will follow from the next

Proposition 2. *Given $L > 1$, consider the set \mathcal{C}_L of all functions $\rho \in C_+(I^2, \mathbb{R})$ such that there is no L -bi-Lipschitz map $\phi : I^2 \rightarrow \mathbb{R}^2$ satisfying $\text{Jac}(\phi) = \rho$ a.e. Then \mathcal{C}_L is an open dense subset of $C_+(I^2, \mathbb{R})$.*

Theorem 1 is a direct consequence of this, since

$$\bigcap_{n \in \mathbb{N}} \mathcal{C}_n = \mathcal{C}.$$

To prove Proposition 2, we first establish that \mathcal{C}_L is an open subset of $C_+(I^2, \mathbb{R})$ for each $L > 0$. Let $X := C_+(I^2, \mathbb{R}) \setminus \mathcal{C}_L$, and consider a sequence $(\rho_k)_{k \in \mathbb{N}} \subset X$ such that $\rho_k \xrightarrow[k \rightarrow \infty]{} \rho$. We need to show that $\rho \notin \mathcal{C}_L$. To do this, let $\phi_k : I^2 \rightarrow \mathbb{R}^2$ be a sequence of L -bi-Lipschitz maps such that $Jac(\phi_k) = \rho_k$ a.e. By Arzelà-Ascoli theorem, $(\phi_k)_{k \in \mathbb{N}}$ has a subsequence that converges to an L -bi-Lipschitz map $\phi : I^2 \rightarrow \mathbb{R}^2$. To simplify notation, we assume without loss of generality that $\phi_k \xrightarrow[k \rightarrow \infty]{} \phi$. Our goal is to show that $Jac(\phi) = \rho$, and hence $\rho \notin \mathcal{C}_L$.

Let $U \subset I^2$ be a closed ball and denote by λ the Lebesgue measure in \mathbb{R}^2 . Then, as $\rho_k \rightarrow \rho$,

$$\begin{aligned} Area(\phi_k(U)) &= \int_U Jac(\phi_k) d\lambda \\ &= \int_U \rho_k d\lambda \\ &\xrightarrow[k \rightarrow \infty]{} \int_U \rho d\lambda. \end{aligned}$$

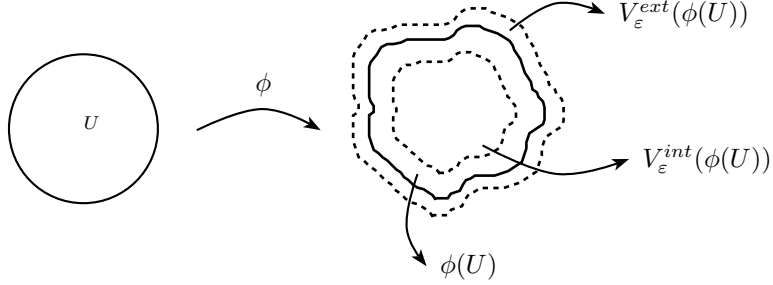


FIGURE 2. The sets $\phi(U)$, $V_\varepsilon^{ext}(\phi(U))$ and $V_\varepsilon^{int}(\phi(U))$.

Below we show that, also, $Area(\phi_k(U)) \rightarrow Area(\phi(U))$, and hence $Jac(\phi) = \rho$, as announced. To do this, given $\varepsilon > 0$, consider the sets

$$V_\varepsilon^{ext}(\phi(U)) := \{x \in \mathbb{R}^2 : d(x, \phi(U)) < \varepsilon\}$$

and

$$V_\varepsilon^{int}(\phi(U)) := \{x \in \phi(U) : d(x, \partial\phi(U)) > \varepsilon\}.$$

Then the desired convergence

$$Area(\phi_k(U)) \rightarrow Area(\phi(U))$$

obviously follows from the next

Lemma 2. *Given $\varepsilon > 0$, there exists a positive integer k_0 such that if $k \geq k_0$, then*

- i) $\phi_k(U) \subset V_\varepsilon^{ext}(\phi(U))$,
- ii) $V_\varepsilon^{int}(\phi(U)) \subset \phi_k(U)$.

Proof. Assume i) does not hold. Then for each $k \in \mathbb{N}$ there exists $y_k \in \phi_k(U) \setminus V_\varepsilon^{ext}(\phi(U))$. Let $x_k \in U$ be such that $y_k = \phi_k(x_k)$. Since U is closed, there exists a

sequence $(x_{k_l})_{l \in \mathbb{N}}$ that converges to a certain $x \in U$. By the equicontinuity of the sequence $(\phi_k)_{k \in \mathbb{N}}$ and the convergences $x_{k_l} \xrightarrow{l \rightarrow \infty} x$ and $\phi_k \rightarrow \phi$, there exists $l_0 \in \mathbb{N}$ such that if $l \geq l_0$, then $|\phi_{k_l}(x_{k_l}) - \phi_{k_l}(x)| < \frac{\varepsilon}{2}$ and $|\phi_{k_l}(x) - \phi(x)| < \frac{\varepsilon}{2}$. Thus, for $l \geq l_0$,

$$|\phi_{k_l}(x_{k_l}) - \phi(x)| \leq |\phi_{k_l}(x_{k_l}) - \phi_{k_l}(x)| + |\phi_{k_l}(x) - \phi(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $y_{k_l} = \phi_{k_l}(x_{k_l}) \xrightarrow{l \rightarrow \infty} \phi(x) \in \phi(U) \subset V_\varepsilon^{\text{ext}}(\phi(U))$. However, this is impossible, since (y_k) is a sequence in the closed subset $\mathbb{R}^2 \setminus V_\varepsilon^{\text{ext}}(\phi(U))$, and then all its accumulation points lie in $\mathbb{R}^2 \setminus V_\varepsilon^{\text{ext}}(\phi(U))$.

To prove ii), let $x, y \in \partial U$ be such that the arc between x and y has length less than $\frac{\varepsilon}{2L}$. Let $k_0 \in \mathbb{N}$ be such that $|\phi_k(x) - \phi(x)| < \varepsilon/4$ and $|\phi_k(y) - \phi(y)| < \varepsilon/4$ for all $k \geq k_0$. Since ϕ_k is L -bi-Lipschitz, the curve in the boundary of $\phi_k(U)$ joining $\phi_k(x)$ and $\phi_k(y)$ has a length $< \varepsilon/2$. Therefore, $\partial(\phi_k(U)) \cap V_\varepsilon^{\text{int}}(\phi(U)) = \emptyset$, and thus $V_\varepsilon^{\text{int}}(\phi(U)) \subset \phi_k(U)$, for $k \geq k_0$. \square

Remark 1. Define $\mathcal{C}_{L,\infty}$ as being the set of all positive functions $\rho \in L_+^\infty(I^2, \mathbb{R})$ such that there is no L -bi-Lipschitz map $f : I^2 \rightarrow \mathbb{R}^2$ satisfying $\text{Jac}(f) = \rho$ a.e. Then the same argument above shows that $\mathcal{C}_{L,\infty}$ is an open subset of $L_+^\infty(I^2, \mathbb{R})$.

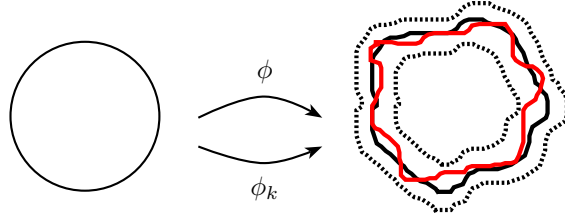


FIGURE 3. The sets $\phi(U)$ (black) and $\phi_k(U)$ (red) for $k \geq k_0$.

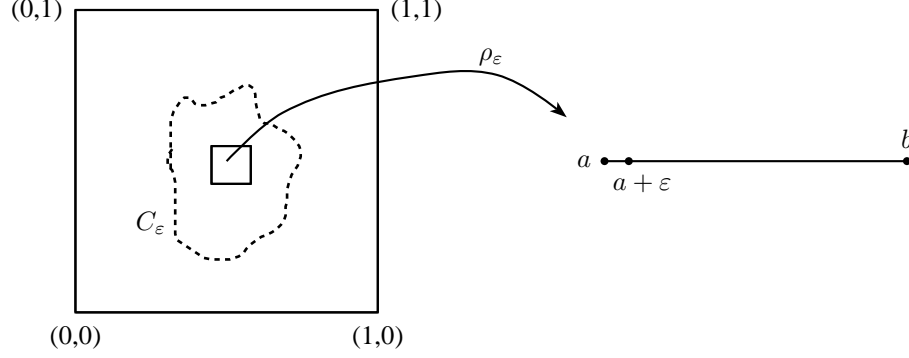
We next establish that \mathcal{C}_L is a dense subset of $C_+(I^2, \mathbb{R})$ for each $L > 0$. Given $\varphi \in C_+(I^2, \mathbb{R})$ with image $[a, b] := \varphi(I^2) \subset (0, \infty)$ and given $\varepsilon > 0$ with $\varepsilon \ll b - a$, we want to construct a continuous function $\rho \in \mathcal{C}_L$ such that $\|\rho - \varphi\|_0 < \varepsilon$. To do this, consider the interval $I_\varepsilon := (a, a + \varepsilon) \subset [a, b]$. Let $C_\varepsilon \subset I^2$ be a connected component of $\varphi^{-1}(I_\varepsilon)$ and S_ε a sufficiently small square contained in C_ε . Let $\rho_\varepsilon : S_\varepsilon \rightarrow I_\varepsilon$ be a continuous density that is not realizable as the Jacobian of an L -bi-Lipschitz homeomorphism (see Figure 4). By the Tietze extension theorem, there exists a continuous function $\hat{\rho}_\varepsilon : C_\varepsilon \rightarrow [a, a + \varepsilon]$ such that $\hat{\rho}_\varepsilon|_{S_\varepsilon} = \rho_\varepsilon$ and

$$\sup_{x \in C_\varepsilon} \hat{\rho}_\varepsilon(x) = \sup_{x \in S_\varepsilon} \rho_\varepsilon(x).$$

Lemma 3. *There exists a continuous function $\rho : I^2 \rightarrow [a, b]$ such that $\rho|_{I^2 \setminus C_\varepsilon} = \varphi$, $\rho|_{S_\varepsilon} = \rho_\varepsilon$, and $\|\rho - \varphi\|_0 < \varepsilon$.*

Proof. Let $\{\varrho_1, \varrho_2\}$ be a partition of unity subordinate to the open cover $\{I^2 \setminus S_\varepsilon, C_\varepsilon\}$ of I^2 . Define $\rho : I^2 \rightarrow [a, b]$ by letting

$$\rho(x) := \varphi \varrho_1 + \hat{\rho}_\varepsilon \varrho_2.$$

FIGURE 4. Construction of the function $\rho_\varepsilon : S_\varepsilon \longrightarrow [a, a + \varepsilon]$.

By definition, it readily follows that $\rho|_{I^2 \setminus C_\varepsilon} = \varphi$ and $\rho|_{S_\varepsilon} = \rho_\varepsilon$. In addition, since $\{\varrho_1, \varrho_2\}$ is a partition of unity, we have

$$\begin{aligned} \|\rho - \varphi\|_0 &= \|\varphi\varrho_1 + \hat{\rho}_\varepsilon\varrho_2 - \varphi\|_0 \\ &= \|\varphi\varrho_1 + \hat{\rho}_\varepsilon\varrho_2 - \varphi(\varrho_1 + \varrho_2)\|_0 \\ &= \|(\hat{\rho}_\varepsilon - \varphi)\varrho_2\|_0. \end{aligned}$$

Now, since $\hat{\rho}_\varepsilon(U_\varepsilon) \subset [a, a + \varepsilon]$, we have $\sup_{x \in C_\varepsilon \setminus S_\varepsilon} |(\hat{\rho}_\varepsilon(x) - \varphi(x))\varphi_2(x)| < \varepsilon$. Therefore, $\|\rho - \varphi\|_0 < \varepsilon$, as desired. \square

4. THE L^∞ CASE

We consider the set $L_+^\infty(I^2, \mathbb{R}) := \{\rho \in L^\infty(I^2, \mathbb{R}) : \rho(x) > 0 \text{ a.e.}\}$. Our goal is to prove the next

Theorem 2. *Let \mathcal{C}_∞ be the set of all positive measurable functions $\rho \in L_+^\infty(I^2, \mathbb{R})$ such that there is no bi-Lipschitz map $f : I^2 \longrightarrow \mathbb{R}^2$ satisfying $Jac(f) = \rho$ a.e. Then \mathcal{C}_∞ is a thick subset of $L_+^\infty(I^2, \mathbb{R})$.*

As in the continuous case, this theorem is a consequence of the next

Proposition 3. *Given $L > 1$, let $\mathcal{C}_{L,\infty}$ be the set of all positive functions $\rho \in L_+^\infty(I^2, \mathbb{R})$ such that there is no L -bi-Lipschitz map $f : I^2 \longrightarrow \mathbb{R}^2$ satisfying $Jac(f) = \rho$ a.e. Then $\mathcal{C}_{L,\infty}$ is an open dense subset of $L_+^\infty(I^2, \mathbb{R})$.*

Proof. By Remark 1, we know that $\mathcal{C}_{L,\infty}$ is open. Let us show that it is a dense subset of $L_+^\infty(I^2, \mathbb{R})$. Given $\varphi \in L_+^\infty(I^2, \mathbb{R})$, denote $b := \sup \varphi(x)$. Fix $\varepsilon > 0$, and define φ_ε as

$$\varphi_\varepsilon(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \geq \varepsilon, \\ \varepsilon & \text{if } \varphi(x) < \varepsilon. \end{cases}$$

Let $a \in [\varepsilon, b]$ be such that $\varphi_\varepsilon^{-1}([a, a + \varepsilon])$ has positive measure. Let y be a Lebesgue density point of $\varphi_\varepsilon^{-1}([a, a + \varepsilon])$, which means that

$$(4.1) \quad \lim_{\delta \rightarrow 0^+} \frac{\lambda(\varphi_\varepsilon^{-1}([a, a + \varepsilon]) \cap S_\delta(y))}{\lambda(S_\delta(y))} = 1,$$

where $S_\delta(y)$ denotes the square centered at y and having sidelength δ . An easy application of Lemma 1 then shows the following: For a small-enough $\delta > 0$, there is a function $\rho_\varepsilon : S_\delta(y) \rightarrow [\varepsilon, b]$ that takes values in $[a, a + \varepsilon]$ for points in $\varphi_\varepsilon^{-1}([a, a + \varepsilon]) \cap S_\delta(y)$ and coincides with φ_ε on the complement of $\varphi_\varepsilon^{-1}([a, a + \varepsilon]) \cap S_\delta(y)$ such that no L -bi-Lipchitz map $f : S_\delta(y) \rightarrow \mathbb{R}^2$ can have a Jacobian equal to ρ_ε . Let ρ be defined by letting

$$\rho(x) = \begin{cases} \rho_\varepsilon(x) & \text{if } x \in S_\delta(y), \\ \varphi_\varepsilon & \text{if } x \notin S_\delta(y). \end{cases}$$

By construction, ρ belongs to $\mathcal{C}_{L,\infty}$, and $\|\rho - \varphi\|_\infty \leq 2\varepsilon$. As this construction can be performed for any $\varepsilon > 0$, this shows that $\mathcal{C}_{L,\infty}$ is a dense subset of $L_+^\infty(I^2, \mathbb{R})$. \square

Acknowledgments. This work was funded by the Fondecyt Project 1160541 and Anillo 1415 (PIA CONICYT) *Geometry at the Frontier*.

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